MATRICES

01. MATRIX

A matrix is an ordered rectangular array of numbers or functions. The numbers or functions are called the elements of the matrix

02. ORDER OF A MATRIX

A matrix having m rows and n columns is called a matrix of order m×n or simply m×n matrix.

or $A=[a_{ij}]_{m\times n}$, $1\le i\le m$, $1\le j\le n$, $i, j\in N$

a_{ij} is an element lying in the i^{th} row & j^{th} column. The number of elements in m×n matrix will be mn.

03. TYPE OF MATRIX

(i) Column Matrix: A matrix is said to be a column matrix if it has only one column, i.e., $A=[a_{ij}]_{m\times 1}$ is a column matrix of order m×1.

(ii) Row Matrix: Row matrix has only one row, i.e., $B=[b_{ij}]_{1\times n}$ is a row matrix of order 1×n.

(iii) Square Matrix: Square matrix has equal number of rows and columns, i.e., $A=[a_{ij}]_{m\times m}$ is a square matrix of order m.

(iv) Diagonal Matrix: A square matrix is said to be diagonal matrix if all of its non-diagonal elements are zero, i.e., $B=[b_{ij}]_{m\times n}$ is said to be a diagonal matrix if $b_{ij}=0$, where $i \ne j$.

(v) Scalar Matrix: It is a diagonal matrix with all its diagonal elements are equal, i.e., $B=[b_{ij}]_{m\times n}$ is a scalar matrix if $b_{ij} = 0$, where $i\ne j$, $b_{ij}=k$, when I = j & k= constant.

(vi) Identity Matrix: : It is a diagonal matrix having all its diagonal elements equal to 1, i.e., $A=[a_{ij}]_{m\times n}$ is an identity matrix if

$$a_{ij}=\begin{cases}1, & \text{if } i=j\\ 0, & \text{if } i\ne j\end{cases}$$

we denote identity matrix by I_n when order is n.

(vii) Zero Matrix: A matrix is said to be zero or null matrix if all its elements are zero. It is denoted by O.

04. EQUALITY OF MATRICES

Two matrices $A=[a_{ij}]$ and $B = [b_{ij}]$ are said to be equal if

(i) they are of the same order

(ii) each element of A is equal to the corresponding element of B, i.e., $a_{ij} = b_{ij}$ for all i & j

05. TRACE OF A MATRIX

The sum of diagonal element of a square matrix A is called the trace of matrix A, which is denoted by tr A

$$\text{tr } A=\sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + \dots a_{nn}$$

Properties of Trace of a Matrix

Let $A = [a_{ij}]_{n\times n}$ and $B = [b_{ij}]$ and λ be a scalar.

(i) $tr(\lambda A) = \lambda tr(A)$ (ii) $tr(A - B)= tr(A) - tr(B)$

(iii) $tr(AB) = tr(BA)$ (iv) $tr(I_n)= n$

(vi) $tr(AB) \ne tr A. tr B$ (v) $tr(O) = 0$

06. ADDITION OF MATRICES

Properties of matrix Addition

(i) Commulative Law: A + B = B + A **(ii) Associative Law:** (A + B) + C = A + (B + C)

(iii) Existence of Additive Identity: Let $A=[a_{ij}]_{m\times n}$ & O = zero matrix of order m × n, then A + O = O + A = A. Here O is the additive identity for matrix addition.

(iv) Existence of Additive Inverse: Let $A= [a_{ij}]_{m\times n}$ be any matrix then we have another matrix as Let $-A= [-a_{ij}]_{m\times n}$ such that A + (-A) = (- A) + A = O. Here -A is the additive inverse of A or negative of A.

07. MULTIPLICATION OF A MATRIX BY A SCALAR

Let $A=[a_{ij}]m\times n$ be a matrix & k, t be a number.

Then, $kA = Ak = [ka_{ij}]_{m\times n}$

Properties

(i) k (A + B) = kA + kB **(ii)** (k + t) A = kA + tA.

08. MULTIPLICATION OF MATRICES

If A & B are any two matrices, then their product will be defined only when the number of columns in A is equal to the number of rows in B

If $A=[a_{ij}]_{m\times n}$ and $B=[b_{ij}]_{n\times p}$ then their product $AB=C=[c_{ij}]$ is a matrix of order, m x p where

$(ij)^{th}$ element of $AB=C_{ij}=\sum_{r=1}^{n} a_{ir}b_{rj}$

09. PROPERTIES OF MATRIX MULTIPLICATON

(i) Associative Law for Multiplication: If A, B & C are three matrices of order $m\times n, n\times p \& p\times q$ respectively, then $(AB)C = A(BC)$

(ii) Distributive Law : For three matrices A, B & C (a) $A(B+C) = AB + AC$
(b) $(A+B)C = AC + BC$ whenever both sides of equality are defined.

(iii) Matrix Multiplication is not commutative in general, i.e. $AB \neq BA$ (in general).

(iv) Existence of Multiplicative Identity: For every square matrix, there exists an identity matrix I of same order such that $IA = AI = A$

10. PROPERTIES OF TRANSPOSE OF THE MATRICES

For any matrices A & B of suitable orders, we have:

(i) $(A^T)^T = A$

(ii) $(kA)^T = k(A)^T$ (where k is constant)

(iii) $(A \pm B)^T = A^T \pm B^T$

(iv) $(AB)^T = B^T A^T$

(v) $(A_1 A_2 A_3 \ldots A_{n-1} A_n)^T = A_n^T A_{n-1}^T \ldots A_3^T A_2^T A_1^T$

(vi) $I^T = I$.

11. MATRIX POLYNOMIAL

Let $f(x) = a_0 x^m + a_1 x^{m-1} + a_2 x^{m-2} + \ldots a_{n-1} x + a_n$ be a polynomial and let A be a square matrix of order n, then $f(A) = a_0 A^m + a_1 A^{m-1} + a_2 A^{m-2} + \ldots + a_{n-1} A + a_n I_n$ is called a matrix polynomial.

12. SYMMETRIC & SKEW SYMMETIC MATRICES

Symmetric Matrix

A square matrix $\mathbf{A} = [a_{ij}]$ is called a symmetric matrix, if $a_{ij} = a_{ji}$ for all i, j or $A^T = A$

Skew Symmetric Matrix

A square matrix $A = [a_{ij}]$ is called a skew-symmetric matrix, if $a_{ij} = -a_{ji}$ for all i, j or $A^T = -A$

Properties of Symmetric & Skew Symmetric Matrices

(i) For any square matrix A with real number entries $(A + A^T)$ is a skew symmetric matrix $(A - \mathbf{A}^T)$ symmetric matrix.

(ii) Any square matrix A can be expressed as the sum of a symmetric & a skew symmetric matrix as $A = \left[\frac{1}{2}(A + A^T)\right] + \left[\frac{1}{2}(A - A^T)\right]$

13. INVERTIBLE MATRIX AND INVERSE MATRIX

Properties of Invertible Matrices

(i) Uniqueness of Inverse : Inverse of a square matrix, if it exists, is unique.

(i) $(A^{-1})^{-1} = A$ (ii) $(A^T)^{-1} = (A^{-1})^T$ (iii) $(AB)^{-1} = B^{-1}A^{-1}$ (iv) $(A^k)^{-1} = (A^{-1})^k$

14. ORTHOGONAL MATRIX

A square matrix A is called orthogonal if $AA^T = I = A^T A$, i.e., if $A^{-1} = A^T$.

Example: $A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} = A^T$. In fact every unit matrix is orthogonal

15. IDEMPOTENT MATRIX

A square matrix A is called an idempotent matrix if $A^2 = A$.

Example: $\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$ is an idempotent matrix, because

$$A^2 = \begin{bmatrix} \frac{1}{4}+\frac{1}{4} & \frac{1}{4}+\frac{1}{4} \\ \frac{1}{4}+\frac{1}{4} & \frac{1}{4}+\frac{1}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = A$$

Also, $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and, $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ are idempotent matrices because $A^2 = A$ and $B^2 = B$.

16. INVOLUTORY MATRIX

A square matrix A is called an involutory matrix if $A^2 = I$ or $A^{-1} = A$

Example: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is an involutory matrix because

$A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ In fact every unit matrix is involutory

17. PERIODIC MATRIX

A matrix A will be called a periodic matrix if $A^{k+1} = A$ where k is a positive integer. If, however k is the least positive integer for which $A^{k+1} = A$, then k is said to be the period of A.

18. NILPOTENT MATRIX

A square matrix A is called a nilpotent matrix if there exists $p \in N$ such that $A^p = 0$.

Example: $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ is a nilpotent matrix because $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$ (Here p = 2)

19. DIFFERENTIATION OF A MATRIX

If $A = \begin{bmatrix} f(x) & g(x) \\ h(x) & l(x) \end{bmatrix}$, then $\frac{dA}{dx} = \begin{bmatrix} f'(x) & g'(x) \\ h'(x) & l'(x) \end{bmatrix}$ is a differentiation of matrix A.